

# Frustration-Induced Gaplessness in the Frustrated Transverse Ising Ring

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New effects in the frustrated transverse Ising ring are predicted. The system is solved based on a mapping of Pauli spin operators to the Jordan-Wigner fermions. We group the low-lying energy levels into bands after imposing appropriate parity constraint, which projects out the redundant degrees of freedom brought about by the Jordan-Wigner fermions. In the region of strong antiferromagnetic coupling, we uncover an unusual gapless phase induced by the ring frustration. We demonstrate that its ground state exhibits a strong longitudinal spin-spin correlation and possesses a considerably large entropy of entanglement. The low-lying energy levels evolve adiabatically in the gapless phase, which facilitates us to work out new behaviors of density of states, low-temperature correlation functions and specific heat. We also propose an experimental protocol for observing this peculiar gapless phase.

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*Introduction.*—Ising chain in transverse field is an excellent prototype for demonstrating quantum phase transition [1]. It can be solved by the Jordan-Wigner transformation, which links the spins and the Jordan-Wigner fermions through a non-local relation [2]. Its theoretical properties have been related to real materials since decades ago [3]. There is a minor mismatch between the Jordan-Wigner fermions and the spins in that the transformation depends on boundary condition [4, 5]. Nevertheless, the discrepancy in the usual problem is negligible and the fermion representation can produce reliable results for the ground state and low excitations. To magnify the mismatch between the fermions and spins, we work on a frustrated transverse Ising ring (FTIR), i.e. the frustrated antiferromagnetic Ising model in a transverse field on a periodic chain with odd number of lattice sites. The model reads

$$H = J \sum_{j=1}^N \sigma_j^x \sigma_{j+1}^x - h \sum_{j=1}^N \sigma_j^z, \quad (1)$$

where the number of lattice sites  $N=2L+1$  ( $L=1,2,3,\dots$ ),  $\sigma_j^\alpha$ 's ( $\alpha=x,z$ ) are Pauli matrices, the longitudinal antiferromagnetic coupling  $J>0$ , the uniform transverse field  $h>0$ . A periodic boundary condition  $\sigma_{N+1}^\alpha = \sigma_1^\alpha$  is imposed. One can choose the representation that ensures the diagonalization of  $\sigma_j^z$ , i.e.  $\sigma_j^z |\uparrow_j\rangle = |\uparrow_j\rangle$  and  $\sigma_j^z |\downarrow_j\rangle = -|\downarrow_j\rangle$ . With  $N$  being odd, the first term in Eq. (1) possesses a ring frustration, i.e. one can not accomplish an alternative arrangement of the two eigenstates of  $\sigma_j^x$ ,  $|\rightarrow_j\rangle = (|\uparrow_j\rangle + |\downarrow_j\rangle)/\sqrt{2}$  and  $|\leftarrow_j\rangle = (|\uparrow_j\rangle - |\downarrow_j\rangle)/\sqrt{2}$  [1], along the chain as shown in Fig. 1.

Frustration is an important topic in quantum magnetism both theoretically and experimentally [6, 7]. How-

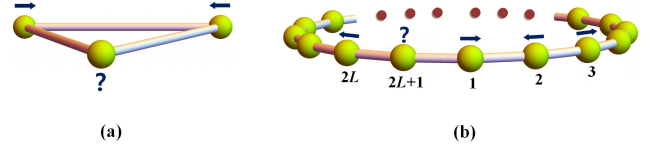


FIG. 1: (Color online) Periodic ring with odd number of sites: (a)  $N=3$ , (b)  $N=2L+1$  for general situation. A ring frustration is induced if we consider the Hamiltonian Eq. (1).

ever exactly solvable model is rare. We found the FTIR exhibits remarkable properties due to the ring frustration despite its simplicity. We will show the frustration induces a peculiar gapless phase in the strong antiferromagnetic coupling region in thermodynamic limit. We analyze significant quantities of the ground state and low-energy excitations to characterize this gapless phase.

*Jordan-Wigner fermions.*—It is convenient to replace the Pauli matrices with the raising and lowering operators  $\sigma_j^\pm = (\sigma_j^x \pm i\sigma_j^y)/2$ . Then one can replace the spin operators with Jordan-Wigner fermions as  $\sigma_1^+ = c_1^\dagger$  and  $\sigma_j^+ = c_j^\dagger \exp(i\pi \sum_{l=1}^{j-1} c_l^\dagger c_l)$  with  $j=2, 3, \dots, N$  [2]. Let us write down the Hamiltonian after the Jordan-Wigner transformation [4],

$$H = Nh - 2h \sum_{j=1}^N c_j^\dagger c_j + J \sum_{j=1}^{N-1} (c_j^\dagger - c_j)(c_{j+1}^\dagger - c_{j+1}) - J \exp(i\pi M)(c_N^\dagger - c_N)(c_1^\dagger + c_1). \quad (2)$$

Although the total number of fermions  $M = \sum_{j=1}^N c_j^\dagger c_j$  does not conserve, the parity  $P = \exp(i\pi M) = (-1)^M$  does. The parity of the system remains to be fixed in con-

text of boundary condition. If neglecting the last term in Eq. (2), one would get a so-called "c-cycle" problem, but we persist on the original "a-cycle" problem here [4]. It is obvious the fermion problem must obey the periodic boundary condition (PBC),  $c_1^\dagger = c_{N+1}^\dagger$ , if  $M$  is an odd number, and the anti-periodic boundary condition (anti-PBC),  $c_1^\dagger = -c_{N+1}^\dagger$ , if  $M$  is an even number [3]. So there are two routes to be followed, which can be called *odd channel (o)* and *even channel (e)* respectively. By introducing a Fourier transformation,  $c_q = (1/\sqrt{N}) \sum_j c_j \exp(iqj)$ , one would find the momentum must take a value in the set  $q^o = \{2m\pi/N | \forall m\}$  for odd channel or  $q^e = \{(2m+1)\pi/N | \forall m\}$  for even channel. In each channel, there are  $N$  available values, i.e.  $m = -(N-1)/2, \dots, -2, -1, 0, 1, 2, \dots, (N-1)/2$ . Then one can diagonalize the Hamiltonian as

$$H^{o/e} = \sum_{q \in q^{o/e}, q \neq q^*} \omega(q) (2\eta_q^\dagger \eta_q - 1) + \epsilon(q^*) (2c_{q^*}^\dagger c_{q^*} - 1) \quad (3)$$

by introducing the Bogoliubov transformation  $\eta_q = u_q c_q - i v_q c_{-q}^\dagger$ , where the coefficients satisfy the relations,  $u_q^2 = (1 + \epsilon(q)/\omega(q))/2$ ,  $v_q^2 = (1 - \epsilon(q)/\omega(q))/2$ ,  $2u_q v_q = \Delta(q)/\omega(q)$ ,  $\omega(q) = \sqrt{\epsilon(q)^2 + \Delta(q)^2}$ ,  $\epsilon(q) = J \cos q - h$ , and  $\Delta(q) = J \sin q$ . Notice we do nothing on the operator  $c_{q^*}$  with  $q^* = 0$  for odd channel and  $q^* = \pi$  for even channel. It will be helpful to introduce a quasi-particle number  $Q = \sum_{q \in (q^o \cup q^e)} n_q$ , where we have defined:  $n_0 = c_0^\dagger c_0$  for  $q = 0$ ,  $n_\pi = c_\pi^\dagger c_\pi$  for  $q = \pi$ , and  $n_q = \eta_q^\dagger \eta_q$  for other  $q$ . It is easy to confirm that  $P = (-1)^M = (-1)^Q$ . The degrees of freedom (DOF) of the fermionic Hamiltonian are  $2^N$  for both channels, so we get  $2^{N+1}$  DOF totally, which is redundantly twice of the DOF of the original spin Hamiltonian. However, the odd channel requires an odd parity and the even channel an even parity. This parity constraint will help us to obliterate exactly the redundant DOF in each channel.

**Bands of energy levels.**—Let us seek for the ground state first. Due to the Bogoliubov transformation, we know the ground state must contain a BCS-like part. Since there are two channels, we have two pure BCS-like functions:

$$|\phi^o\rangle = \prod_{q \in q^o, q \neq 0} (u_q + i v_q c_q^\dagger c_{-q}^\dagger) |0\rangle, \quad (4)$$

$$|\phi^e\rangle = \prod_{q \in q^e, q \neq \pi} (u_q + i v_q c_q^\dagger c_{-q}^\dagger) |0\rangle. \quad (5)$$

It is easy to check that the state  $|\phi^o\rangle$  does not satisfy the parity constraint of the odd channel and should be left out. While  $|\phi^e\rangle$  is a valid state, however, it possesses too high energy to become the ground state. The true ground state comes from the odd channel and can be easily constructed by  $|\phi^o\rangle$  as

$$|E_0\rangle = c_0^\dagger |\phi^o\rangle, \quad (6)$$

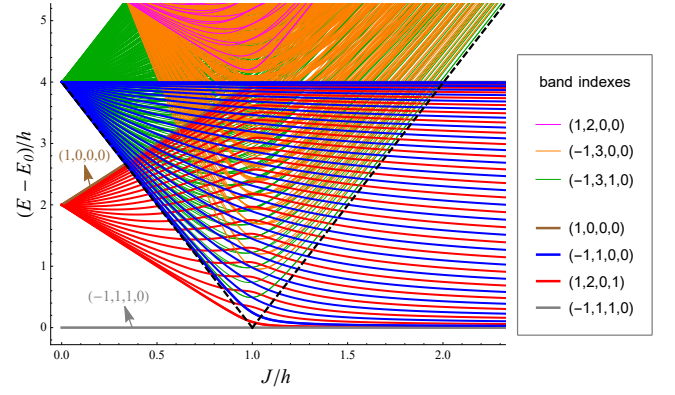


FIG. 2: (Color online) Bands of low energy levels of the frustrated transverse Ising ring. We take  $N = 51$  for a graphical view. The non-degenerate ground state with band indexes  $(P, Q, n_0, n_\pi) = (-1, 1, 1, 0)$  is set as the reference. The dashed black line denotes the lower bound of all bands other than the lowest four ones listed in the right panel. Its intersecting point at  $J/h = 1$  is a critical point in thermodynamic limit, whose right-hand side will become a gapless region.

whose energy reads  $E_0 = \Lambda^o + 2\omega(0)\theta(J/h - 1)$  where  $\Lambda^o = -\sum_{q \in q^o} \omega(q)$  and  $\theta(x)$  is a Heaviside step function. In thermodynamic limit  $N \rightarrow \infty$ , the ground state energy per site reads  $E_0/N = \frac{2}{\pi} |J - h| E(-4Jh/(J-h)^2) + 2(J-h)\theta(J/h - 1)/N$ , where  $E(x)$  is the complete elliptic integral of the second kind. It is non-analytic at  $J/h = 1$ , because its second derivative in respect of  $J/h$  has a logarithmic divergent peak  $\sim (1/\pi) \ln |J/h - 1|$ , which heralds a critical point. In fact, the self-duality [8] of FTIR still holds, which ensures this critical point. One can see this clear by defining new Ising-type operators,  $\tau_j^z = -\sigma_j^x \sigma_{j+1}^x$  and  $\tau_j^x = (-1)^j \prod_{k < j} \sigma_k^z$ , to get a dual form of Hamiltonian,  $H = -J \sum_{j=1}^N \tau_j^z + h \sum_{j=1}^{N-1} \tau_j^x \tau_{j+1}^x$ .

Above the ground state, there is an energy gap  $\Delta_{gap} = 2(h - J)$  to the first excited states for  $J/h < 1$ , which is similar to the usual case [1]. While for  $J/h > 1$ , the system becomes gapless surprisingly. This unexpected gapless phase is the main focus of this Letter. As shown in Fig. 2, there are  $2N$  levels involved in the low-energy properties. All these  $2N$  levels can be uniquely designated by  $q$  and constructed by  $\eta_q^\dagger$  in regard of the parity of the channel. They can be grouped into four bands that are labelled by a set of indexes  $(P, Q, n_0, n_\pi)$ , because the levels with the same set of indexes are degenerate at  $J=0$  and emanate to form a band with  $J$  increasing. The lowest level  $|E_0\rangle$  with indexes  $(-1, 1, 1, 0)$  and the upper-most level  $|\phi^e\rangle$  with indexes  $(1, 0, 0, 0)$  are non-degenerate (we shall label it as  $|E_\pi\rangle = |\phi^e\rangle$ ). Between them, the levels from even and odd channels deplete the width of  $4h$  with alternatively increasing energies. Each level is doubly degenerate. The levels from even channel are  $|E_q\rangle = \eta_q^\dagger c_\pi^\dagger |\phi^e\rangle$

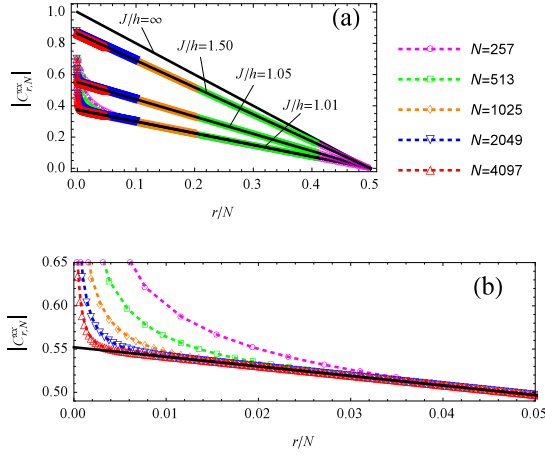


FIG. 3: (Color online) Longitudinal correlation functions for several selected values of  $J/h$  in the gapless phase. In (a), the colored dingbat data are numerical results from the determinant Eq. (9), while the black straight lines are analytical results from Eq. (12). (b) is a zoom-in plot for the case  $J/h = 1.05$ .

( $q \in q^e, q \neq \pi$ ) with indexes (1,2,0,1), whose energy reads  $E_q = \Lambda^e + 2\omega(q)$  with  $\Lambda^e = -\sum_{q \in q^e} \omega(q)$ . While the ones from odd channel are  $|E_q\rangle = \eta_q^\dagger |\phi^o\rangle$  ( $q \in q^o, q \neq 0$ ) with indexes (-1,1,0,0), whose energy reads  $E_q = \Lambda^o + 2\omega(q)$ . We have  $\Lambda = \Lambda^o = \Lambda^e$  if  $N$  is large enough.

*Correlation function of the ground state.*—The two-point longitudinal spin-spin correlation function of the ground state is defined as  $C_{r,N}^{xx} = \langle E_0 | \sigma_j^x \sigma_{j+r}^x | E_0 \rangle$ , where we notice the result does not depend on  $j$ . If we introduce the operators,  $A_j = c_j^\dagger + c_j$  and  $B_j = c_j^\dagger - c_j$ , and use the relations,  $A_j^2 = 1$  and  $A_j B_j = \exp(-i\pi c_j^\dagger c_j)$ , we have

$$C_{r,N}^{xx} = \langle \phi^o | c_0 B_j A_{j+1} B_{j+1} \dots B_{j+r-1} A_{j+r} c_0^\dagger | \phi^o \rangle. \quad (7)$$

To evaluate this function thoroughly, we need to make use of the Wick's theorem and the contractions [4] in respect of  $|\phi^o\rangle$ :  $\langle c_0 c_0^\dagger \rangle = 1$ ,  $\langle A_j c_0^\dagger \rangle = 1/\sqrt{N}$ ,  $\langle B_j c_0^\dagger \rangle = -1/\sqrt{N}$ ,  $\langle A_j A_{j+r} \rangle = \delta_{r,0}$ ,  $\langle B_j B_{j+r} \rangle = -\delta_{r,0}$ , and  $\langle B_j A_{j+r} \rangle = D_{r+1}$  with

$$D_r = \frac{1}{N} \sum_{q \in q^o, q \neq 0} \exp(-iqr) D(e^{iq}) - \frac{1}{N}, \quad (8)$$

where  $D(e^{iq}) = -(J - h e^{iq})/\omega(q)$ . Then we arrive at a Toeplitz determinant

$$C_{r,N}^{xx} = \begin{vmatrix} D_0 + \frac{2}{N} & D_{-1} + \frac{2}{N} & \dots & D_{-r+1} + \frac{2}{N} \\ D_1 + \frac{2}{N} & D_0 + \frac{2}{N} & \dots & D_{-r+2} + \frac{2}{N} \\ \dots & \dots & \dots & \dots \\ D_{r-1} + \frac{2}{N} & D_{r-2} + \frac{2}{N} & \dots & D_0 + \frac{2}{N} \end{vmatrix}. \quad (9)$$

For finite  $N$ , this determinant can be evaluated numerically. While for  $N \rightarrow \infty$ , we need to substitute the sum

$\frac{1}{N} \sum_{q \in q^o, q \neq 0}$  in Eq. (8) with an integral  $\int \frac{dq}{2\pi}$  by defining appropriate  $D(e^{iq})$  to evaluate the determinant analytically [9].

First, let us focus on the most intriguing gapless phase ( $J/h > 1$ ). Following the earlier procedure of Wu [9–11], we put forward a proposition [12]:

*Proposition:* Consider a Toeplitz determinant

$$\Theta(r, N, x, e^{ik}) = \begin{vmatrix} \tilde{D}_0 & \tilde{D}_{-1} & \dots & \tilde{D}_{1-r} \\ \tilde{D}_1 & \tilde{D}_0 & \dots & \tilde{D}_{2-r} \\ \dots & \dots & \dots & \dots \\ \tilde{D}_{r-1} & \tilde{D}_{r-2} & \dots & \tilde{D}_0 \end{vmatrix} \quad (10)$$

with  $\tilde{D}_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} D(e^{iq}) e^{-iqn} + \frac{x}{N} e^{ikn}$ . If the generating function  $D(e^{iq})$  and  $\ln D(e^{iq})$  are continuous on the unit circle  $|e^{iq}| = 1$ , then the behavior for large  $N$  of  $\Theta(r, N, x, e^{ik})$  is given by ( $1 \ll r \ll N$ )

$$\Theta(r, N, x, e^{ik}) = \Delta_r (1 + \frac{xr}{ND(e^{-ik})}), \quad (11)$$

where  $\Delta_r = \mu^r \exp(\sum_{n=1}^{\infty} n d_{-n} d_n)$ ,  $\mu = \exp[\int_{-\pi}^{\pi} \frac{dq}{2\pi} \ln D(e^{iq})]$ , and  $d_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-iqn} \ln D(e^{iq})$ , if the sum  $\sum_{n=1}^{\infty} n d_{-n} d_n$  is convergent.

By applying this proposition to the gapless phase ( $x=2$  and  $k=0$ ), we get

$$C_{r,N}^{xx} \approx (-1)^r (1 - \frac{h^2}{J^2})^{1/4} (1 - \frac{2r}{N}) \quad (12)$$

for large enough  $r$  and  $N$ . This asymptotic behaviour is depicted in Fig. 3, which is perfectly coincident with the numerical results. For  $h = 0$ , Eq. (12) becomes rigorous. This surprising result is totally different from the conventional case without frustration [1], and needs a further elucidation.

While for the gapped phase ( $J/h < 1$ ), the Toeplitz determinant recovers the usual case without frustration [9]. The resulting correlation function decays exponentially with a finite correlation length  $\xi = -1/\ln(J/h)$  [3, 13]. And for the critical point ( $J/h=1$  and  $N \rightarrow \infty$ ), the correlation function decays algebraically as  $C_{r,\infty}^{xx} \sim (-1)^r \alpha / r^{1/4}$  [9]. Our numerical results are in good agreement with these conclusions, which will not be presented here since they are not the focus of this Letter.

*Entanglement entropy of the ground state.*—The entanglement entropy (EE) is defined as the von Neumann entropy  $S_l = -\text{tr}(\rho_l \log_2 \rho_l)$  of the reduced density matrix  $\rho_l = \text{tr}_{N-l} |E_0\rangle\langle E_0|$ , where the trace is performed on the spin states of contiguous sites from  $j=1$  to  $N-l$  [14]. We can evaluate the EE numerically by utilizing the matrix

$$\Gamma_l = \begin{vmatrix} \Pi_0 & \Pi_1 & \dots & \Pi_{l-1} \\ \Pi_{-1} & \Pi_0 & \dots & \Pi_{l-2} \\ \dots & \dots & \dots & \dots \\ \Pi_{1-l} & \Pi_{2-l} & \dots & \Pi_0 \end{vmatrix} \quad \text{with } \Pi_l = \begin{vmatrix} 0 & -g_l \\ g_{-l} & 0 \end{vmatrix}, \quad (13)$$

where  $g_l = D_{l-1} + \frac{2}{N}$ . Let  $V \in SO(2l)$  denote an orthogonal matrix that brings  $\Gamma_l$  into a block diagonal form such that  $\Gamma_l^C = V \Gamma_l V^T = \bigoplus_{m=0}^{l-1} (iv_m \sigma_y)$  with  $v_m \geq 0$ . Then  $S_l$  is given by  $S_l = \sum_{m=0}^{l-1} H_2(\frac{1+v_m}{2})$  with  $H_2(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ . The numerical results for  $l = (N-1)/2$  are shown in Fig. 4. We observe the EE of the gapped phase is small until near the critical point, where it abruptly tends to become divergent as predicted by CFT [15]. While in the gapless phase, we observe  $S_{(N-1)/2}$  with  $N \rightarrow \infty$  approaches its minimal value 2 when  $h \rightarrow 0$ , where the FTIR reduces to the classical Ising model with the ring frustration,  $H^{\text{Ising}} = J \sum_{j=1}^N \sigma_j^x \sigma_{j+1}^x$ . The EE of the gapless phase is much larger than that of the well-known GHZ states, whose EE  $S_{(N-1)/2}$  is exactly  $\log_2 2 = 1$  [14]. The  $2N$ -degenerate ground states of  $H^{\text{Ising}}$  can be denoted by kink states [16]:

$$\begin{aligned} |K(j), \rightarrow\rangle &= |\cdots, \leftarrow_{j-1}, \rightarrow_j, \rightarrow_{j+1}, \leftarrow_{j+2}, \cdots\rangle, \\ |K(j), \leftarrow\rangle &= |\cdots, \rightarrow_{j-1}, \leftarrow_j, \leftarrow_{j+1}, \rightarrow_{j+2}, \cdots\rangle, \end{aligned} \quad (14)$$

where kinks occur between sites  $j$  and  $j+1$ . The kink states break parity symmetry. One can verify the ground state  $|E_0\rangle$  with odd parity evolves adiabatically into a simple sum of the kink states when  $h \rightarrow 0$ :

$$\lim_{h \rightarrow 0} |E_0\rangle = \frac{1}{\sqrt{2N}} \sum_{j,\tau} |K(j), \tau\rangle, \quad (15)$$

whose EE is exactly 2 in thermodynamic limit. In fact, all of the degenerate  $2N$  states near  $h \sim 0$  are just recombinations of the kink states. We usually take it for granted that the classical system  $H^{\text{Ising}}$  falls into one of the kink states at zero temperature due to spontaneous symmetry breaking. But the  $2N$  levels split into bands when quantum fluctuations arise with  $h > 0$ , which prevents the spontaneous symmetry breaking and thus protects the odd parity of the ground state.

*Finite temperature properties of the gapless phase.*— In the gapless phase, the lowest  $2N$  neatly aligned levels dominate the system's properties at low temperatures ( $T \ll 4h/k_B$ ), where  $k_B$  is the Boltzmann constant. This fact facilitates us to work out some quantities at low temperatures based on the partition function  $Z = \sum_{q \in q^o \text{ and } q^e} e^{-\beta E_q}$ , where  $\beta = \frac{1}{k_B T}$ . The density of states (DOS) is defined as  $\rho(x) = \frac{1}{N} \sum_{q \in q^o \text{ and } q^e} \delta(E - E_q)$ . Considering the thermodynamic limit, we get a DOS at low energy sector

$$\rho(x) = \frac{4(x+2J-2h)}{\pi \sqrt{x(x-4h)(4h-x-4J)(x+4J)}}, \quad (16)$$

which can be expanded as  $\rho(x) = ax^{-1/2} + bx^{1/2} + O(x^{3/2})$  with  $x = E - E_0$ ,  $a = \frac{(J-h)^{1/2}}{\pi(Jh)^{1/2}}$ , and  $b = \frac{(h^2+Jh+J^2)}{8\pi(Jh)^{3/2}(J-h)^{1/2}}$ . So we get the specific heat per site at low temperatures,

$$\frac{C_M(T)}{N} = \frac{k_B}{2} \left[ 1 + \frac{2bk_B T(4a + bk_B T)}{(2a + bk_B T)^2} \right], \quad (17)$$

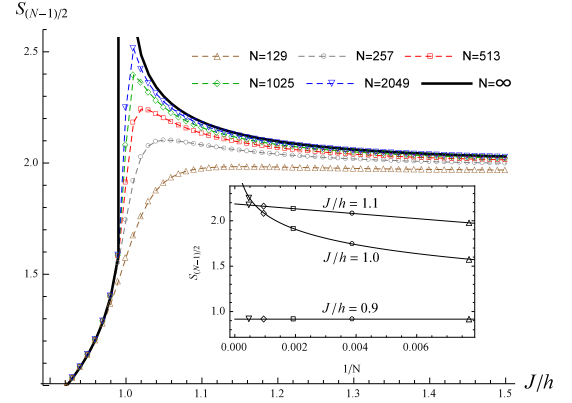


FIG. 4: (Color online) Entanglement entropy  $S_{(N-1)/2}$  as a function of  $J/h$  for a sequence of number of lattice sites  $N$ . The inset shows examples of finite size scaling for extrapolating to the thermodynamic limit. At the critical point, the numerical data fit a divergent behavior,  $S_{(N-1)/2} \sim \frac{1}{6} \log_2 N$ , coincident with the prediction by CFT [15].

which reaches a constant  $k_B/2$  at zero temperature. Next, we consider the correlation function at low temperatures. When we evaluated the correlation function for each of the lowest  $2N$  levels, we got an unexpected finding:  $\langle E_q | \sigma_j^x \sigma_{j+r}^x | E_q \rangle = \langle E_0 | \sigma_j^x \sigma_{j+r}^x | E_0 \rangle$  for  $q \in q^o$  and  $q^e$ , although the resulting Toeplitz determinants are a little different [12]. So we draw a conclusion that the correlation function for large enough  $r$  and  $N$  is inert to the change of temperature at low temperatures,

$$C_{r,N}^{xx}(0 \leq T \ll 4h/k_B) = C_{r,N}^{xx}. \quad (18)$$

*Experimental proposal.*—Although it is not possible to get an infinite FTIR system to observe its fascinating properties, we may design a large enough one to see the trend and realize the fascinating states within nowadays state-of-art techniques based on laser-cooled and trapped atomic ions. In fact, the case for  $N = 3$  has been experimentally realized [17]. To generate a system with larger  $N$ , we provide another proposal.

In our proposal as shown in Fig. 5, the key point is to produce a ring geometrical potential with odd number of traps. In  $x$ - $y$  plane, we need to impose  $N$  (odd) beams of independent standing wave lasers which are obtained by frequency selection. Then, each standing wave will contributes an optical potential along  $\vec{k}_i$  direction that can be expressed as  $V_{x-y} \cos^2(\vec{k}_i \cdot \vec{r}_i - \phi)$  for the  $i$ -th beam, where  $\vec{k}_i$  is the strength of beams and  $\phi$  is the phase shift. The angle between two neighboring lasers is  $2\pi/N$ . Thus, by adopting  $V_{x-y}$  and  $\phi$ , we obtain a circular lattice potential with  $N$  traps in  $x$ - $y$  plane. In  $z$  direction we apply two independent standing wave lasers,  $V_{z1} \cos^2(k_z z)$  and  $V_{z2} \cos^2(2k_z z)$ , where the former has twice wave length of the latter. Eventually, we obtain a



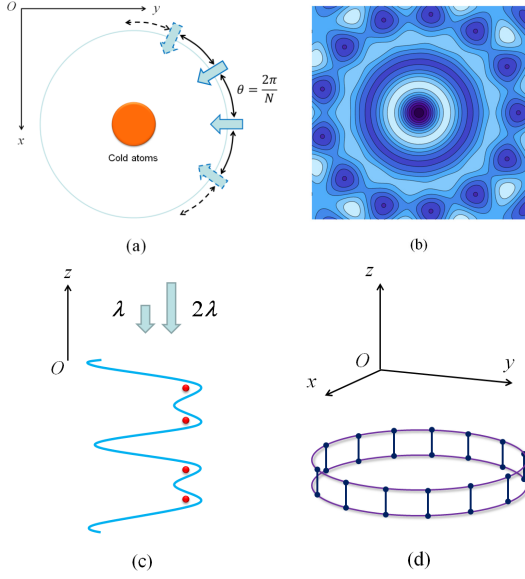


FIG. 5: (Color online) (a) Scheme of the proposed experimental setup in  $x$ - $y$  plane. Each arrow depicts a wave vector of a standing-wave laser. The angle between any two neighboring lasers is  $2\pi/N$ . (b) The exemplified color map of optical potential where a ring of 13 trapping wells is shown by the dark blue potential wells. (c) The arrangement of lasers in  $z$  direction, where two standing wave lasers form an isolated double wells potential. (d) The total two-leg ladder potential.

periodical two-leg ladder potential by forming a double-well potential in  $z$  direction.

Next, we consider loading into the ladders with cold atoms having two relevant internal states. For sufficiently deep potential and low temperatures, the system will be described by a bosonic or fermionic Hubbard model [18]. In the Mott-insulating phase, it can be reduced to a pseudo-spin  $XXZ$  model by second-order perturbation at half-filling. By modulating the intensity and the phase shift of the trapping laser beams and by adjusting scattering length through Feshbach resonance, we can eventually arrive at the pseudo-spin Hamiltonian [12],

$$H_s = \sum_{j,s=1,2} J_z S_{j,s}^z S_{j+1,s}^z + \sum_j K \vec{S}_{j,1} \cdot \vec{S}_{j,2}, \quad (19)$$

The low-energy properties of this system are dominated by the pseudo-spin singlet  $|s\rangle_j = (|\uparrow\downarrow\rangle_j - |\downarrow\uparrow\rangle_j)/\sqrt{2}$  and triplet  $|t_0\rangle_j = (|\uparrow\downarrow\rangle_j + |\downarrow\uparrow\rangle_j)/\sqrt{2}$  on the rung of the ladders in low energy. At this time, the system can be mapped exactly to an effective FTIR [19].

**Conclusion.**—In summary, we have shown an unusual gapless phase induced by the ring frustration in the FTIR. We demonstrated a novel behavior of the correlation function in the gapless phase at both zero and low temperatures. The ground state in the gapless phase is dominated by a superposition of all possible antiferromagnetic kink states, which results in a quite large

entropy of entanglement. The non-degeneracy of the ground state prevents spontaneous symmetry breaking. These unexpected outcomes are in close relation to the oddity of the number of lattice sites even in thermodynamic limit. If given an even number of lattice sites, the gapless phase would turn into a gapped one as the usual schematic way [1]. The oddity-induced phenomenon revealed in this Letter is reminiscent of the one in the well-known spin ladders [20]. We also proposed an experimental protocol for observing the peculiar gapless phase.

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## Supplemental material

### I. The correlation function in the gapless phase

#### Proposition and proof

*Proposition:* Consider a Toeplitz determinant

$$\Theta(r, N, x, e^{ik}) = \begin{vmatrix} \tilde{D}_0 & \tilde{D}_{-1} & \cdots & \tilde{D}_{1-r} \\ \tilde{D}_1 & \tilde{D}_0 & \cdots & \tilde{D}_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{D}_{r-1} & \tilde{D}_{r-2} & \cdots & \tilde{D}_0 \end{vmatrix} \quad (\text{S1})$$

with  $\tilde{D}_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} D(e^{iq}) e^{-iqn} + \frac{x}{N} e^{ikn}$ . If the generating function  $D(e^{iq})$  and  $\ln D(e^{iq})$  are continuous on the unit circle  $|e^{iq}| = 1$ , then the behavior for large  $N$  of  $\Theta(r, N, x, e^{ik})$  is given by ( $1 \ll r \ll N$ )

$$\Theta(r, N, x, e^{ik}) = \Delta_r (1 + \frac{xr}{ND(e^{-ik})}), \quad (\text{S2})$$

where

$$\Delta_r = \mu^r \exp\left(\sum_{n=1}^{\infty} n d_{-n} d_n\right), \quad (\text{S3})$$

$$\mu = \exp\left[\int_{-\pi}^{\pi} \frac{dq}{2\pi} \ln D(e^{iq})\right], \quad (\text{S4})$$

$$d_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-iqn} \ln D(e^{iq}), \quad (\text{S5})$$

if the sum in Eq. (S3) converges.

*Proof:* Let  $e^{iq} = \xi$ ,  $D_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} D(\xi) \xi^{-n}$ , then  $\tilde{D}_n = D_n + \frac{x}{N} e^{ikn}$ . First, we rewrite Eq. (S1) as

$$\begin{aligned} \Theta(r, N, x, e^{ik}) = & \begin{vmatrix} D_0 & D_{-1} & \cdots & D_{-r+1} \\ D_1 & D_0 & \cdots & D_{-r+2} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & D_{r-2} & \cdots & D_0 \end{vmatrix} + \begin{vmatrix} \frac{x}{N} & D_{-1} & \cdots & D_{1-r} \\ \frac{x}{N} e^{ik} & D_0 & \cdots & D_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{x}{N} e^{i(r-1)k} & D_{r-2} & \cdots & D_0 \end{vmatrix} \\ & + \cdots + \begin{vmatrix} D_0 & \frac{x}{N} e^{-ik} & \cdots & D_{2-r} \\ D_1 & \frac{x}{N} & \cdots & D_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & \frac{x}{N} e^{i(r-2)k} & \cdots & D_0 \end{vmatrix} + \begin{vmatrix} D_0 & D_{-1} & \cdots & \frac{x}{N} e^{i(1-r)k} \\ D_1 & D_0 & \cdots & \frac{x}{N} e^{i(2-r)k} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & D_{r-2} & \cdots & \frac{x}{N} \end{vmatrix}. \end{aligned}$$

Then we compose a set of linear equations

$$\sum_{m=0}^{r-1} D_{n-m} x_m^{(r-1)} = \frac{x}{N} e^{ikn}, \quad 0 \leq n \leq r-1. \quad (\text{S6})$$

These equations have an unique solution for  $x_n^{(r-1)}$  if there exists a non-zero determinant:

$$\Delta_r \equiv \begin{vmatrix} D_0 & D_{-1} & \cdots & D_{1-r} \\ D_1 & D_0 & \cdots & D_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & D_{r-2} & \cdots & D_0 \end{vmatrix} \neq 0. \quad (\text{S7})$$

By Cramer's rule, we have the solution:

$$x_0^{(r-1)} = \frac{\begin{vmatrix} \frac{x}{N} & D_{-1} & \cdots & D_{1-r} \\ \frac{x}{N} e^{ik} & D_0 & \cdots & D_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{x}{N} e^{i(r-1)k} & D_{r-2} & \cdots & D_0 \end{vmatrix}}{\Delta_r}, \quad (\text{S8})$$

$$x_1^{(r-1)} = \frac{\begin{vmatrix} D_0 & \frac{x}{N} e^{-ik} & \cdots & D_{2-r} \\ D_1 & \frac{x}{N} & \cdots & D_{2-r} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & \frac{x}{N} e^{i(r-2)k} & \cdots & D_0 \end{vmatrix}}{\Delta_r}, \quad (\text{S9})$$

$$\vdots$$

$$x_{r-1}^{(r-1)} = \frac{\begin{vmatrix} D_0 & D_{-1} & \cdots & \frac{x}{N} e^{i(1-r)k} \\ D_1 & D_0 & \cdots & \frac{x}{N} e^{i(2-r)k} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r-1} & D_{r-2} & \cdots & \frac{x}{N} \end{vmatrix}}{\Delta_r}. \quad (\text{S10})$$

So we arrive at

$$\Theta(r, N, x, e^{ik}) = \Delta_r + \Delta_r \sum_{n=0}^{r-1} e^{-ikn} x_n^{(r-1)}. \quad (\text{S11})$$

For our problem,  $\Delta_r$  can be evaluated directly by using Szegő's theorem, so we need to know how to calculate the second term in Eq. (S11). Follow the standard Wiener-Hopf procedure [9–11], we consider a generalization of Eq. (S6)

$$\sum_{m=0}^{r-1} D_{n-m} x_m = y_n, \quad 0 \leq n \leq r-1 \quad (\text{S12})$$

and define

$$x_n = y_n = 0 \quad \text{for } n \leq -1 \quad \text{and } n \geq r \quad (\text{S13})$$

$$\begin{aligned} v_n &= \sum_{m=0}^{r-1} D_{-n-m} x_m \quad \text{for } n \geq 1 \\ &= 0 \quad \text{for } n \leq 0 \end{aligned} \quad (\text{S14})$$

$$\begin{aligned} u_n &= \sum_{m=0}^{r-1} D_{r-1+n-m} x_m \quad \text{for } n \geq 1 \\ &= 0 \quad \text{for } n \leq 0 \end{aligned} \quad (\text{S15})$$

We further define

$$D(\xi) = \sum_{n=-\infty}^{\infty} D_n \xi^n, \quad Y(\xi) = \sum_{n=0}^{r-1} y_n \xi^n, \quad V(\xi) = \sum_{n=1}^{\infty} v_n \xi^n, \quad U(\xi) = \sum_{n=1}^{\infty} u_n \xi^n, \quad X(\xi) = \sum_{n=0}^{r-1} x_n \xi^n. \quad (\text{S16})$$

It then follows from Eq. (S12) that we can get

$$D(\xi) X(\xi) = Y(\xi) + V(\xi^{-1}) + U(\xi) \xi^{r-1} \quad (\text{S17})$$

for  $|\xi| = 1$ . Because  $D(\xi)$  and  $\ln D(\xi)$  is continuous and periodic on the unit circle,  $D(\xi)$  has a unique factorization, up to a multiplicative constant, in the form

$$D(\xi) = P^{-1}(\xi) Q^{-1}(\xi^{-1}), \quad (\text{S18})$$

for  $|\xi| = 1$ , such that  $P(\xi)$  and  $Q(\xi)$  are both analytic for  $|\xi| < 1$  and continuous and nonzero for  $|\xi| \leq 1$ . we may now use the factorization of  $D(\xi)$  in Eq. (S17) to write

$$\begin{aligned} P^{-1}(\xi) X(\xi) - [Q(\xi^{-1}) Y(\xi)]_+ - [Q(\xi^{-1}) U(\xi) \xi^{r-1}]_+ \\ = [Q(\xi^{-1}) Y(\xi)]_- + Q(\xi^{-1}) V(\xi^{-1}) + [Q(\xi^{-1}) U(\xi) \xi^{r-1}]_-, \end{aligned} \quad (\text{S19})$$

where the subscript  $+$  ( $-$ ) means that we should expand the quantity in the brackets into a Laurent series and keep only those terms where  $\xi$  is raised to a non-negative (negative) power. The left-hand side of Eq. (S19) defines a function analytic for  $|\xi| < 1$  and continuous on  $|\xi| = 1$  and the right-hand side defines a function which is analytic for  $|\xi| > 1$  and is continuous for  $|\xi| = 1$ . Taken together they define a function  $E(\xi)$  analytic for all  $\xi$  except possibly for  $|\xi| = 1$  and continuous everywhere. But these properties are sufficient to prove that  $E(\xi)$  is an entire function which vanished at  $|\xi| = \infty$  and thus, by Liouville's theorem, must be zero everywhere [9, 10]. Therefore both the right-hand side and the left-hand side of Eq. (S19) vanish separately and thus we have

$$X(\xi) = P(\xi) \left\{ [Q(\xi^{-1}) Y(\xi)]_+ + [Q(\xi^{-1}) U(\xi) \xi^{r-1}]_+ \right\}. \quad (\text{S20})$$

Furthermore,  $U(\xi)$  can be neglected for large  $r$

$$X(\xi) \approx P(\xi) [Q(\xi^{-1}) Y(\xi)]_+. \quad (\text{S21})$$

Consider the term  $[Q(\xi^{-1}) Y(\xi)]_+$ , because  $Q(\xi)$  is a  $+$  function, so we can expand it as a Laurent series and keep only those term where  $\xi$  is raised to a non-negative power,

$$Q(\xi) = \sum_{n=0}^{\infty} a_n \xi^n = (a_0 + a_1 \xi^1 + a_2 \xi^2 + \dots + a_{r-1} \xi^{r-1}) + O(\xi^r), \quad (\text{S22})$$

and then

$$Q(\xi^{-1}) = a_0 + a_1 \xi^{-1} + a_2 \xi^{-2} + \dots + a_{r-1} \xi^{1-r}, \quad (\text{S23})$$

where we have neglected the term  $O(\xi^r)$  for large  $r$  for clarity, from Eq. (S6) and Eq. (S16), we have

$$Y(\xi) = \sum_{n=0}^{r-1} y_n \xi^n = \frac{x}{N} \left( 1 + e^{ik} \xi^1 + e^{2ik} \xi^2 + \dots + e^{i(r-1)k} \xi^{r-1} \right), \quad (\text{S24})$$

thus

$$\begin{aligned} [Q(\xi^{-1}) Y(\xi)]_+ = \frac{x}{N} \left[ (a_0 + a_1 e^{ik} + a_2 e^{2ik} + \dots + a_{r-1} e^{i(r-1)k}) \right. \\ \left. + (a_0 e^{ik} + a_1 e^{2ik} + \dots + a_{r-2} e^{i(r-1)k}) \xi^1 + \dots + (a_0 e^{i(r-1)k}) \xi^{r-1} \right]. \end{aligned} \quad (\text{S25})$$

From Eq. (S11), Eq. (S16) and Eq. (S21), we have

$$\sum_{n=0}^{r-1} e^{-ikn} x_n^{(r-1)} = X(e^{-ik}) = P(e^{-ik}) [Q(e^{ik}) Y(e^{-ik})]_+, \quad (\text{S26})$$

$$\begin{aligned} [Q(e^{ik}) Y(e^{-ik})]_+ = \frac{x}{N} \left[ r a_0 + r a_1 e^{ik} + r a_2 e^{2ik} + \dots + r a_{r-1} e^{i(r-1)k} \right] \\ - \frac{x}{N} \left[ a_1 e^{ik} + 2 a_2 e^{2ik} + \dots + (r-1) a_{r-1} e^{i(r-1)k} \right] \\ = \frac{x}{N} \left[ r Q(e^{ik}) - e^{ik} \frac{dQ(\xi)}{d\xi} \Big|_{\xi=e^{ik}} \right]. \end{aligned} \quad (\text{S27})$$



So when  $r \gg 1$ , we can ignore the second term in Eq. (S27). Together with Eq. (S18), we get

$$X(e^{-ik}) = \frac{xr}{N} P(e^{-ik}) Q(e^{ik}) = \frac{xr}{ND(e^{-ik})}. \quad (\text{S28})$$

At last, by Szegő's theorem, we get

$$\Delta_r = \mu^r \exp\left(\sum_{n=1}^{\infty} n d_{-n} d_n\right), \quad (\text{S29})$$

where

$$\begin{aligned} \mu &= \exp\left[\int_{-\pi}^{\pi} \frac{dq}{2\pi} \ln D(e^{iq})\right], \\ d_n &= \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-iqn} \ln D(e^{iq}). \end{aligned} \quad (\text{S30})$$

From Eq. (S11), we have

$$\Theta(r, N, x, e^{ik}) = \Delta_r \left(1 + \frac{xr}{ND(e^{-ik})}\right). \quad (\text{S31})$$

Q.E.D.

### Application to the FTIR: the correlation functions

We now specialize to the problem of the FTIR in the main text. We calculate the correlation functions of the lowest  $2N$  levels that are grouped into four bands. The first band with indexes  $(-1, 1, 1, 0)$  is the ground state coming from the odd channel,  $|E_0\rangle = c_0^\dagger |\phi^o\rangle$ . We have

$$\begin{aligned} C_{r,N}^{x,x}(|E_0\rangle) &= \langle \phi^o | c_0 B_j A_{j+1} B_{j+1} \dots B_{j+r-1} A_{j+r} c_0^\dagger | \phi^o \rangle \\ &= \begin{vmatrix} D_0 + \frac{2}{N} & D_{-1} + \frac{2}{N} & \dots & D_{1-r} + \frac{2}{N} \\ D_1 + \frac{2}{N} & D_0 + \frac{2}{N} & \dots & D_{2-r} + \frac{2}{N} \\ \dots & \dots & \dots & \dots \\ D_{r-1} + \frac{2}{N} & D_{r-2} + \frac{2}{N} & \dots & D_0 + \frac{2}{N} \end{vmatrix}, \end{aligned} \quad (\text{S32})$$

where

$$D_n = \frac{1}{N} \sum_{q \in q^o, q \neq 0} \exp(iq(n-1)) (1 - 2u_q^2 - 2iu_q v_q) - \frac{1}{N}. \quad (\text{S33})$$

The second band with indexes  $(1, 2, 0, 1)$  is a collection of states from the even channel,  $|E_k\rangle = \eta_k^\dagger c_\pi^\dagger |\phi^e\rangle$  ( $k \in q^e$ ,  $k \neq \pi$ ). We have

$$\begin{aligned} C_{r,N}^{x,x}(|E_k\rangle) &= \frac{1}{2} \left( \langle \phi^e | c_\pi \eta_k B_j A_{j+1} \dots B_{j+r-1} A_{j+r} \eta_k^\dagger c_\pi^\dagger | \phi^e \rangle + \langle \phi^e | c_\pi \eta_{-k} B_j A_{j+1} \dots B_{j+r-1} A_{j+r} \eta_{-k}^\dagger c_\pi^\dagger | \phi^e \rangle \right) \\ &= \frac{1}{2} [\Gamma^e(r, N, \alpha_k, e^{ik}) + \Gamma^e(r, N, \alpha_{-k}, e^{-ik})], \end{aligned} \quad (\text{S34})$$

where

$$\Gamma^e(r, N, \alpha_k, e^{ik}) = \begin{vmatrix} F_0 + \frac{2\alpha_k}{N} & F_{-1} + \frac{2\alpha_k}{N} e^{-ik} & \dots & F_{1-r} + \frac{2\alpha_k}{N} e^{i(1-r)k} \\ F_1 + \frac{2\alpha_k}{N} e^{ik} & F_0 + \frac{2\alpha_k}{N} & \dots & F_{2-r} + \frac{2\alpha_k}{N} e^{i(2-r)k} \\ \dots & \dots & \dots & \dots \\ F_{r-1} + \frac{2\alpha_k}{N} e^{i(r-1)k} & F_{r-2} + \frac{2\alpha_k}{N} e^{i(r-2)k} & \dots & F_0 + \frac{2\alpha_k}{N} \end{vmatrix}, \quad (\text{S35})$$

$$\langle \phi^e | B_l A_m | \phi^e \rangle = F_{l-m+1} = \frac{1}{N} \sum_{q \in q^e} \exp(iq(l-m)) (1 - 2u_q^2 - 2iu_q v_q). \quad (\text{S36})$$

$$\alpha_k = \frac{J - h e^{-i k}}{\sqrt{(J - h e^{-i k})(J - h e^{i k})}}. \quad (\text{S37})$$

Notice that we have defined  $u_\pi^2 = 0, 2u_\pi v_\pi = 0$  in Eq. (S36). The third band with indexes  $(-1, 1, 0, 0)$  is a collection of states from the odd channel,  $|E_k\rangle = \eta_k^\dagger |\phi^o\rangle (k \in q^o, k \neq 0)$ . We have

$$\begin{aligned} C_{r,N}^{x,x}(|E_k\rangle) &= \frac{1}{2} \left( \langle \phi^o | \eta_k B_j A_{j+1} \dots B_{j+r-1} A_{j+r} \eta_k^\dagger | \phi^o \rangle + \langle \phi^o | \eta_{-k} B_j A_{j+1} \dots B_{j+r-1} A_{j+r} \eta_{-k}^\dagger | \phi^o \rangle \right) \\ &= \frac{1}{2} [\Gamma^o(r, N, \alpha_k, e^{i k}) + \Gamma^o(r, N, \alpha_{-k}, e^{-i k})], \end{aligned} \quad (\text{S38})$$

where

$$\Gamma^o(r, N, \alpha_k, e^{i k}) = \begin{vmatrix} D_0 + \frac{2\alpha^k}{N} & D_{-1} + \frac{2\alpha^k}{N} e^{-i k} & \dots & D_{1-r} + \frac{2\alpha^k}{N} e^{i(1-r)k} \\ D_1 + \frac{2\alpha^k}{N} e^{i k} & D_0 + \frac{2\alpha^k}{N} & \dots & D_{2-r} + \frac{2\alpha^k}{N} e^{i(2-r)k} \\ \dots & \dots & \dots & \dots \\ D_{r-1} + \frac{2\alpha^k}{N} e^{i(r-1)k} & D_{r-2} + \frac{2\alpha^k}{N} e^{i(r-2)k} & \dots & D_0 + \frac{2\alpha^k}{N} \end{vmatrix}, \quad (\text{S39})$$

with  $\alpha_k$  ( $k \in q^o, k \neq 0$ ) given by Eq. (S37). The last band with indexes  $(1, 0, 0, 0)$  is the state from the even channel,  $|E_\pi\rangle = |\phi^e\rangle$ . We have

$$\begin{aligned} C_{r,N}^{x,x}(|E_\pi\rangle) &= \langle \phi^e | B_j A_{j+1} B_{j+1} \dots A_{j+r-1} B_{j+r-1} A_{j+r} | \phi^e \rangle \\ &= \begin{vmatrix} F_0 + \frac{2}{N} & F_{-1} + \frac{2}{N} e^{-i\pi} & \dots & F_{1-r} + \frac{2}{N} e^{i\pi(1-r)} \\ F_1 + \frac{2}{N} e^{i\pi} & F_0 + \frac{2}{N} & \dots & F_{2-r} + \frac{2}{N} e^{i\pi(2-r)} \\ \dots & \dots & \dots & \dots \\ F_{r-1} + \frac{2}{N} e^{i\pi(r-1)} & F_{r-2} + \frac{2}{N} e^{i\pi(r-2)} & \dots & F_0 + \frac{2}{N} \end{vmatrix}, \end{aligned} \quad (\text{S40})$$

Note that Eq. (S32), Eq. (S34), Eq. (S38) and Eq. (S40) are valid for the gapped phase, the critical point and the gapless phase. For the most interesting gapless phase, we have  $u_0^2 = 1, 2u_0 v_0 = 0$ . we can rewrite Eq. (S33) as

$$D_n = \frac{1}{N} \sum_{q \in q^o} \exp(i q (n-1)) (1 - 2u_q^2 - 2i u_q v_q). \quad (\text{S41})$$

For large  $N$ , we have

$$D_n = F_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-i q n} \frac{-(J - h e^{i q})}{\sqrt{(J - h e^{-i q})(J - h e^{i q})}}. \quad (\text{S42})$$

So the above four cases of correlation function in the gapless phase can be written in an uniform formula

$$C_{r,N}^{x,x}(|E_q\rangle) = \frac{1}{2} [\Gamma(r, N, \alpha_k, e^{i k}) + \Gamma(r, N, \alpha_{-k}, e^{-i k})], \quad (\text{S43})$$

where

$$\Gamma(r, N, \alpha_k, e^{i k}) = \begin{vmatrix} \tilde{D}_0 & \tilde{D}_{-1} & \dots & \tilde{D}_{1-r} \\ \tilde{D}_1 & \tilde{D}_0 & \dots & \tilde{D}_{2-r} \\ \dots & \dots & \dots & \dots \\ \tilde{D}_{r-1} & \tilde{D}_{r-2} & \dots & \tilde{D}_0 \end{vmatrix}, \quad (\text{S44})$$

$$\tilde{D}_n = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-i q n} \frac{-(J - h e^{i q})}{\sqrt{(J - h e^{-i q})(J - h e^{i q})}} + \frac{2\alpha^k}{N} e^{i k n},$$

$$\alpha^k = \frac{J - h e^{-i k}}{\sqrt{(J - h e^{-i k})(J - h e^{i k})}}, \quad \text{with} \quad k \in \begin{cases} k^o = -\frac{(N-1)\pi}{N}, \dots, -\frac{2\pi}{N}, 0, \frac{2\pi}{N}, \dots, \frac{(N-1)\pi}{N} & \text{for odd channel} \\ k^e = -\frac{(N-2)\pi}{N}, \dots, -\frac{\pi}{N}, \frac{\pi}{N}, \dots, \frac{(N-2)\pi}{N}, \pi & \text{for even channel} \end{cases} \quad (\text{S45})$$

We can evaluate  $\Gamma(r, N, \alpha_k, e^{ik})$  by applying the *Proposition* directly to get

$$\Gamma(r, N, \alpha_k, e^{ik}) = (-1)^r \left(1 - \frac{h^2}{J^2}\right)^{1/4} \left(1 - \frac{2r}{N}\right) = \Gamma(r, N, \alpha_{-k}, e^{-ik}), \quad (\text{S46})$$

for large  $r$  and  $N$ . So the asymptotic behavior of the correlation functions is given by

$$C_{r,N}^{x,x}(|E_q\rangle) = (-1)^r \left(1 - \frac{h^2}{J^2}\right)^{1/4} \left(1 - \frac{2r}{N}\right), \quad (\text{S47})$$

which is independent of  $q$ . Thus, the correlation function is not sensitive to the temperature if the temperature is low enough, i.e.

$$C_{r,N}^{xx}(0 \leq T \ll 4h/k_B) = (-1)^r \left(1 - \frac{h^2}{J^2}\right)^{1/4} \left(1 - \frac{2r}{N}\right). \quad (\text{S48})$$

## II. Experimental proposal

First, we propose how to construct a ring-geometric optical lattice potential containing  $N$  trapping wells. We only focus on the cases where  $N$  is an odd number, i.e.  $N = 2L + 1$  ( $L = 1, 2, 3, \dots$ ). In  $x$ - $y$  plane we can arrange  $N$  beams of independent standing wave lasers which can be obtained by frequency selection. Each standing wave will contribute an optical potential along  $\vec{k}_i$  direction which can be expressed as  $V_{x-y} \cos^2(\vec{k}_i \cdot \vec{r}_i - \phi)$  for  $i$ -th laser beam.  $V_{x-y}$  is the strength of laser beams and  $\phi$  is the phase shift. The angle between two neighbor lasers is  $2\pi/N$  [Fig. 5(a) in the main text]. Then, by adopting  $V_{x-y}$  and  $\phi$ , we find that we can always obtain a circular lattice potential with  $N$  traps in  $x$ - $y$  plane [Fig. 5(b) of the main text]. As shown in Fig. 5(c) in the main text, in  $z$  direction we apply two independent standing wave lasers  $V_{z1} \cos^2(k_z z)$  and  $V_{z2} \cos^2(2k_z z)$  where one laser has twice wave length of the second laser. The potential along  $z$  direction has a double well shape. By adopting the relative strengths  $V_{z1}$  and  $V_{z2}$  of the two lasers, we can neglect the hopping and interaction of cold atoms between inter-double-well. Eventually, we obtain a periodical two-leg ladder potential shown in Fig. 5(d) of the main text. In real experiment, there is additional harmonic trapping potential  $V_{trap}(x^2 + y^2)$ . The total potential can be written as

$$V_{potential}(x, y, z) = V_{trap}(x^2 + y^2) + V_{z1} \cos^2(k_z z) + V_{z2} \cos^2(2k_z z) + V_{x-y} \sum_{i=1}^N \cos^2(\vec{k}_i \cdot \vec{r}_i - \phi). \quad (\text{S49})$$

Second, we discuss the scheme to realize the transverse Ising model. Let us consider loading into the ladders with cold atoms having two relevant internal states that are denoted as pseudo-spin states  $\lambda = \uparrow, \downarrow$ . The lattice potential experienced by cold atoms depends on which of those two internal states are located. For sufficiently deep potential and low temperatures, the system will be described by the following bosonic or fermionic Hubbard model [18],

$$H_{Hub} = \sum_{j,\lambda,s} (-t_\lambda) (a_{j\lambda,s}^\dagger a_{(j+1)\lambda,s} + \text{h.c.}) + \sum_{j,\lambda} (-t_\lambda) (a_{j\lambda,1}^\dagger a_{j\lambda,2} + \text{h.c.}) + \frac{1}{2} \sum_{j,\lambda,s} U_\lambda n_{j\lambda,s} (n_{j\lambda,s} - 1) + \sum_{j,s} U_{\uparrow\downarrow} n_{j\uparrow,s} n_{j\downarrow,s}, \quad (\text{S50})$$

where  $s = 1, 2$  is the leg index. With the conditions of Mott insulator limit  $t_\lambda \ll U_\lambda$ ,  $U_{\uparrow\downarrow}$  and half filling  $\langle n_{j\uparrow,s} \rangle + \langle n_{j\downarrow,s} \rangle \approx 1$ , the low-energy Hamiltonian of Eq. (S50) is mapped to the spin  $XXZ$  model by second-order perturbation,

$$H_{spin} = \sum_{j,s} \pm J_\perp (S_{j,s}^x S_{j+1,s}^x + S_{j,s}^y S_{j+1,s}^y) + J_z S_{j,s}^z S_{j+1,s}^z + \sum_j \pm K_\perp (S_{j,1}^x S_{j,2}^x + S_{j,1}^y S_{j,2}^y) + K_z S_{j,1}^z S_{j,2}^z, \quad (\text{S51})$$

where the pseudo-spin operator  $\mathbf{S} = a^\dagger \vec{\lambda} a / 2$ ,  $\vec{\lambda} = (\lambda_x, \lambda_y, \lambda_z)$  are the Pauli matrices and  $a^\dagger = (a_\uparrow^\dagger, a_\downarrow^\dagger)$ . The positive signs before  $J_\perp, K_\perp$  are for fermionic atoms and negative signs for bosonic one. The interaction coefficients for bosons are given by,

$$J_\perp = \frac{4t_\uparrow t_\downarrow}{U_{\uparrow\downarrow}}, J_z = \frac{2(t_\uparrow^2 + t_\downarrow^2)}{U_{\uparrow\downarrow}} - \frac{t_\uparrow^2}{U_\uparrow} - \frac{t_\downarrow^2}{U_\downarrow}, K_\perp = \frac{4t'_\uparrow t'_\downarrow}{U_{\uparrow\downarrow}}, K_z = \frac{2t'^2_\uparrow + t'^2_\downarrow}{U_{\uparrow\downarrow}} - \frac{t'^2_\uparrow}{U_\uparrow} - \frac{t'^2_\downarrow}{U_\downarrow}. \quad (\text{S52})$$

For fermions, we only need to omit the last two terms in  $J_z$  and  $K_z$ . By modulating the intensity and the phase shift of the trapping laser beams and by adjusting scattering length through Feshbach resonance, we can obtain a desired Hamiltonian from Eq. (S51),

$$H_s = \sum_{j,s} J_z S_{j,s}^z S_{j+1,s}^z + \sum_j K \vec{S}_{j,1} \cdot \vec{S}_{j,2}. \quad (\text{S53})$$

The properties of this system are dominated by the pseudo-spin singlet  $|s\rangle_j = (|\uparrow\downarrow\rangle_j - |\downarrow\uparrow\rangle_j)/\sqrt{2}$  and triplet  $|t_0\rangle_j = (|\uparrow\downarrow\rangle_j + |\downarrow\uparrow\rangle_j)/\sqrt{2}$  on the rung of the ladders in low energy. At this time, the system can be mapped exactly to an effective FTIR [19],

$$H_T = \sum_j 2J_z \tilde{S}_j^x \tilde{S}_{j+1}^x - K \tilde{S}_j^z. \quad (\text{S54})$$